PARTIAL REGULARITY AND AMPLITUDE

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ABSTRACT. The author continues the study of Frobenius amplitude, introduced in an earlier paper, and compares this to various other positivity notions. These are applied to deduce vanishing theorems. A refinement of Castelnuovo-Mumford regularity is also given.

In [A2], this author defined the *Frobenius amplitude* $\phi(\mathcal{E})$ of an algebraic vector bundle $\mathcal{E}$. This is an integer which is defined by reduction modulo p and which provides a measure of the positivity of $\mathcal{E}$. This notion was introduced in order to formulate, and prove, the following vanishing theorem: on a smooth projective variety X defined over a field of characteristic 0, $H^q(X, \Omega_X^p \otimes \mathcal{E}) = 0$ for $p + q > \dim X + \phi(\mathcal{E})$. The main goal of this paper is to demonstrate the efficacy of this result (and its refinements) by extracting generalizations of a number of known theorems. The point is to replace the Frobenius amplitude by a more accessible expression involving the *amplitude* $\alpha(\mathcal{E})$, which is defined below. The basic result proved in section 6, which subsumes the vanishing theorems of Le Potier, Sommese and others, is that the above groups vanish for $p + q > \dim X + rank(\mathcal{E}) + \alpha(\mathcal{E})$ when $\mathcal{E}$ satisfies some additional assumptions.

Over a field of characteristic $p > 0$, the Frobenius powers $\mathcal{E}^{(p^n)}$ of a vector bundle $\mathcal{E}$ can be defined by raising its transition matrices to p^n. The Frobenius amplitude $\phi(\mathcal{E})$ is the threshold for which $H^i(\mathcal{E}^{(p^n)} \otimes \mathcal{F}) = 0$ for any coherent $\mathcal{F}$, $n \gg 0$ and $i > \phi(\mathcal{E})$. In general, the definition is indirect since it involves specialization into positive characteristic. Replacing Frobenius powers by symmetric powers in the above definition, leads to the amplitude $\alpha(\mathcal{E})$ of $\mathcal{E}$. While this invariant is probably new, it is closely related to a number of preexisting positivity notions. For example, the amplitude of a vector bundle is at most k if it is k-ample in Sommese's sense, or if its dual is strongly $k+1$-convex in Andreotti and Grauert's sense when the field is $\mathbb{C}$. The technical heart of this paper is section 5, where the bounds given in [A2, theorems 6.1, 6.7, 7.1] are refined by using amplitude in the place of ampleness. In particular, the key inequality $\phi(\mathcal{E}) \leq rank(\mathcal{E}) + \alpha(\mathcal{E})$ is obtained under appropriate assumptions.

The notion of partial Castelnuovo-Mumford regularity arose as a technical tool in the course of carrying out the above program. However, it gradually became clear that this concept is interesting on its own. In rough terms, for partial regularity the conditions in Mumford's definition of 0-regularity are required to hold for cohomology groups in degrees greater than a number called the *level* of the sheaf. The key point is that, unlike the notion of amplitude, the level involves a finite number of conditions, so it is easier to control in families. The level is 0 precisely for 0-regular sheaves. Several well known results about regular sheaves generalize quite nicely. For example, the fact that the tensor product of regular vector bundles on $\mathbb{P}^n$ is

Partially supported by the NSF.

regular, generalizes to the subadditivity of the level under tensor products. The main ingredient in the proof of this and related properties is the Beilinson-Orlov ([B], [Or]) resolution of the diagonal.

In the final section, I consider the special class of varieties which admit endomorphisms of degree prime to the characteristic. These include Abelian and toric varieties. For such varieties a vanishing theorem is obtained which is much stronger than anything available for general varieties. The result generalizes known facts about Abelian and toric varieties going back to Mumford and Danilov. The proofs, however, have an entirely different character; they are modeled on Frobenius splitting arguments [MR], with the endomorphism playing the role of the Frobenius.

I would like to thank A. Dhillon for numerous discussions about varieties with endomorphisms, Y. Kawamata, D. Keeler and K. Matsuki for various helpful conversations and emails, and the referee for catching a number of errors.

1. Partial regularity

Let k be a field. Recall that a coherent sheaf $\mathcal{F}$ on $\mathbb{P}^n_k$ is 0-*regular*, or simply just regular, if $H^i(\mathcal{F}(-i)) = 0$ for all $i > 0$ [Mu1]. We measure the deviation from regularity by defining the *level* of $\mathcal{F}$ to be

$$\lambda(\mathcal{F}) = \max(\{q \mid \exists i \geq 0, H^{q+i}(\mathbb{P}^n, \mathcal{F}(-1-i)) \neq 0\} \cup \{0\})$$

Equivalently, $\lambda(\mathcal{F})$ is the smallest natural number for which $q > \lambda(\mathcal{F})$ implies

$$H^q(\mathcal{F}(-1)) = H^{q+1}(\mathcal{F}(-2)) = \ldots = 0$$

It follows that $\lambda(\mathcal{F}) = 0$ if and only if $\mathcal{F}$ is regular. An alternative description of the level is that it measures the complexity of Beilinson's spectral sequence (remark 1.10). The point is that regularity is known to be equivalent to this spectral sequence collapsing to a single row concentrated along the horizontal axis [C].

Lemma 1.1. *If*

$$0 \to \mathcal{E}_1 \to \mathcal{E}_2 \to \mathcal{E}_3 \to 0$$

is an extension, then $\lambda(\mathcal{E}_2) \leq max(\lambda(\mathcal{E}_1), \lambda(\mathcal{E}_3))$.

Proof. This should be clear from the definition of λ and the exact sequence for cohomology. □

Let X be a smooth projective variety defined over k. Fix a very ample line bundle $\mathcal{L}$ on X, and set $O_X(1) = \mathcal{L}^{\otimes N}$ for some $N > 0$ to be specified later. The level of a sheaf on X will be measured using $O_X(1)$. Let $A = \oplus H^0(X, O_X(i))$ denote the projective coordinate ring. The vector spaces B_m are defined inductively by $B_0 = k$, $B_1 = H^0(X, O_X(1))$ and

$$B_m = ker[B_{m-1} \otimes H^0(X, O_X(1)) \to B_{m-2} \otimes H^0(X, O_X(2))]$$

We can produce a complex of graded A-modules

$$B_M \otimes A(-M+1) \to \ldots B_1 \otimes A(-1) \to A \to k \to 0 \tag{1}$$

Set $\mathcal{R}_0 = O_X$, and

$$\mathcal{R}_m = ker[B_m \otimes O_X \to B_{m-1} \otimes O_X(1)]$$

for $m > 0$. From (1), we obtain a complex of sheaves

$$0 \to \mathcal{R}_m \otimes_{O_X} O_X(-m+1) \to B_m \otimes_k O_X(-m+1) \to \ldots B_1 \otimes_k O_X \to O_X(1) \to 0 \tag{2}$$

Theorem 1.2 (Orlov)**.** *Let $O_X(1) = \mathcal{L}^{\otimes N}$ as above. Fix $M \geq m > 0$. Then, there is a complex of sheaves on $X \times X$*

$$\mathcal{R}_M \boxtimes O_X(-M) \to \dots \mathcal{R}_1 \boxtimes O_X(-1) \to \mathcal{R}_0 \boxtimes O_X \to O_\Delta \to 0 \tag{3}$$

where $\Delta \subset X \times X$ is the diagonal. The complexes (1), (2) and (3) are exact for $N \gg 0$.

Proof. Orlov [Or, prop A.1] shows that the obstructions to exactness of the above complexes lie in a finite number of coherent cohomology groups, which vanish for large N by Serre. □

Remark 1.3. *We are attributing this result to Orlov, since his paper is (as far as we know) the first place where it has been stated in this form. However there is some closely related prior work that ought to be mentioned. The ring A has been shown to be Koszul for large N by Backelin* [Ba]*, and this implies the exactness of (1) and (3) for all M* [Ka, section 3]*. The theorem can also be deduced from the work of Ein and Lazarsfeld* [EL]*.*

Standard semicontinuity and generic flatness arguments yield:

Corollary 1.4. *Let $\mathcal{X} \to T$ be a flat family over an irreducible quasicompact base, and let $\mathcal{L}$ be a relatively very ample line bundle on $\mathcal{X}$. For any $M \geq m > 0$, there exists an N and a nonempty open set $U \subset T$ such that for $O_{\mathcal{X}}(1) = \mathcal{L}^{\otimes N}$ the complexes (1), (2) and (3) are exact on all the fibers $\mathcal{X}_t$ over the closed points $t \in U$.*

We will say that $O_X(1)$ is *sufficiently ample* if the complexes (1), (2) and (3) are for exact $M = \dim X$. Orlov's theorem guarantees that a high power of a fixed ample bundle is sufficiently ample, and that this true for all nearby fibers if X varies in a family.

The fundamental example is the following.

Example 1.5. *Let $X = \mathbb{P}^n$. The hyperplane bundle $O_X(1)$ is sufficiently ample. The coordinate ring $A \cong k[x_0, \dots x_n]$. The complex (1) is just the Koszul resolution of $A/(x_0, \dots x_n)$, $\mathcal{R}_m \cong \Omega^m(m)$, and (3) is Beilinson's resolution* [B]*:*

$$0 \to \Omega^n(n) \boxtimes O(-n) \to \dots \Omega^1(1) \boxtimes O(-1) \to O \boxtimes O$$

of O_Δ.

This resolution is the basis for Beilinson's spectral sequences. One of which is:

$$E_1^{ab} = H^b(\mathbb{P}^n, \Omega^{-a}(-a) \otimes \mathcal{E}) \otimes O(a) \Rightarrow \mathcal{E} \tag{4}$$

Theorem 1.2 allows us to construct a similar spectral sequence for any X.

Lemma 1.6. *Suppose that $O_X(1)$ is sufficiently ample. Let $\mathcal{E}$ and $\mathcal{F}$ be coherent sheaves with one of them locally free. Then if for each pair of integers $a \geq 0$ and $b \geq 0$, one of $H^b(\mathcal{E} \otimes \mathcal{R}_a) = 0$ or $H^{i+a-b}(\mathcal{F}(-a)) = 0$, then*

$$H^i(\mathcal{E} \otimes \mathcal{F}) = 0$$

Proof. Tensoring Orlov's resolution (3) with $\mathcal{E} \boxtimes \mathcal{F}$ yields an exact sequence

$$\dots (\mathcal{E} \otimes \mathcal{R}_1) \boxtimes \mathcal{F}(-1) \to (\mathcal{E} \otimes \mathcal{R}_0) \boxtimes \mathcal{F} \to \mathcal{E} \otimes \mathcal{F} \to 0$$

Breaking this into short exact sequences and writing out the long exact sequences for cohomology (or by using a spectral sequence), we obtain $H^i(\mathcal{E}\otimes\mathcal{F}) = 0$ provided that

$$H^{i+a}(\mathcal{E}\otimes\mathcal{R}_a \boxtimes \mathcal{F}(-a)) = 0$$

for all $a \geq 0$. The lemma now follows from Künneth's formula. □

We can control the cohomology groups in the preceding lemma by the following.

Lemma 1.7. *Suppose that $O_X(1)$ is sufficiently ample. For any coherent sheaf $\mathcal{F}$ and $a \geq 0$, $H^j(X, \mathcal{R}_i \otimes \mathcal{F}(-a)) = 0$ for $j > \lambda(\mathcal{F}) + a$ and $0 \leq i \leq \dim X$.*

Proof. We prove this by descending induction on a, starting from $a = \dim X$ where it is trivial. The sequence (2) yields

$$0 \to \mathcal{R}_{i+1} \otimes O_X(-1) \to B_{i+1} \otimes O_X(-1) \to \mathcal{R}_i \to 0$$

This implies

$$B_{i+1} \otimes H^j(\mathcal{F}(-a-1)) \to H^j(X, \mathcal{R}_i \otimes \mathcal{F}(-a)) \to H^{j+1}(X, \mathcal{R}_{i+1} \otimes \mathcal{F}(-a-1))$$

and the lemma follows. □

Corollary 1.8. $H^j(X, \mathcal{F}(-a)) = 0$ *for* $j > \lambda(\mathcal{F}) + a$.

Corollary 1.9. $H^j(X, \mathcal{R}_i \otimes \mathcal{F}) = 0$ *for* $j > \lambda(\mathcal{F})$.

Remark 1.10. *The last corollary implies that there are no nonzero rows of Beilinson's spectral sequence (4) other than the first λ rows.*

Proposition 1.11. *Suppose that $O_X(1)$ is chosen sufficiently ample, and suppose that $\mathcal{E}$ and $\mathcal{F}$ are two coherent sheaves on X with $\mathcal{E}$ locally free. Then $\lambda(\mathcal{E}\otimes\mathcal{F}) \leq \lambda(\mathcal{E}) + \lambda(\mathcal{F})$*

Proof. It suffices to prove that

$$H^{q+i}(\mathcal{E}\otimes\mathcal{F}(-i-1)) = 0 \tag{5}$$

for $q > \lambda(\mathcal{E})+\lambda(\mathcal{F})$ and $i \geq 0$. Given $b \geq 0$, we must have $b > \lambda(\mathcal{E})$ or $q-b > \lambda(\mathcal{F})$. In the first case, $H^b(\mathcal{E}\otimes\mathcal{R}_a) = 0$ by corollary 1.9. In the second case,

$$H^{q+i+a-b}(\mathcal{F}(-i-a-1)) = 0.$$

Therefore, lemma 1.6 implies that (5) holds. □

Recall that a coherent sheaf $\mathcal{F}$ is called m-regular if $\mathcal{F}(m)$ is regular. The regularity $reg(\mathcal{F})$ is the smallest m such that $\mathcal{F}(m)$ is regular. We note that $H^i(\mathcal{F}) = 0$ for $m \geq reg(\mathcal{F})$ and $i > 0$ [Mu1, p. 100].

The following result is well known for $\mathbb{P}^n$ [L].

Corollary 1.12. *Given $\mathcal{E}, \mathcal{F}$ and $O_X(1)$ as above. If $\mathcal{E}$ is p-regular and $\mathcal{F}$ is q-regular, then $\mathcal{E}\otimes\mathcal{F}$ is $p+q$-regular.*

Proof. By assumption,

$$\lambda(\mathcal{E}\otimes\mathcal{F}(p+q)) \leq \lambda(\mathcal{E}(p)) + \lambda(\mathcal{F}(q)) = 0.$$

□

Lemma 1.13. *If $\mathcal{E}$ and $\mathcal{F}$ are as above. Then $\lambda(\mathcal{E}\otimes\mathcal{F}) \leq \lambda(\mathcal{E}(-reg(\mathcal{F})))$ and $H^i(\mathcal{E}\otimes\mathcal{F}) = 0$ for $i > \lambda(\mathcal{E}(-reg(\mathcal{F})))$.*

Proof. Let $r = reg(\mathcal{F})$. Proposition 1.11 implies

$$\lambda(\mathcal{E} \otimes \mathcal{F}) = \lambda(\mathcal{E}(-r) \otimes \mathcal{F}(r)) \leq \lambda(\mathcal{E}(-r)) + 0$$

Corollary 1.8 shows that $H^i(\mathcal{E} \otimes \mathcal{F}) = 0$ for $i > \lambda(\mathcal{E}(-r)) \geq \lambda(\mathcal{E} \otimes \mathcal{F})$ □

2. Amplitude

Let k be a field, and let X be a projective variety defined over k. Let $Vect(r) = Vect(X, r)$ be the set of isomorphism classes of vector bundles of rank r over X, and let $Vect(X) = \cup_r Vect(X, r)$. Let $P^N : Vect(X) \to Vect(X), N \in \mathbb{N}$ be a sequence of operations. We define the *P-amplitude* $\alpha_P(\mathcal{E})$ of a vector bundle $\mathcal{E}$ to be the smallest integer i_0 such that for every locally free sheaf $\mathcal{F}$ there exists an N_0 such that $H^i(X, P^N(\mathcal{E}) \otimes \mathcal{F}) = 0$ for $i > i_0$ and $N \geq N_0$. Of particular interest are the *amplitude* $\alpha(\mathcal{E})$ and the *$\otimes$-amplitude* $\alpha_\otimes(\mathcal{E})$, where the operations $P^\bullet$ are the symmetric and tensor powers respectively. If $char\, k = p > 0$, we also have the Frobenius amplitude ϕ where $P^\bullet$ are the Frobenius powers. In more explicit terms, these are the smallest integers for which

$$H^i(X, S^N(\mathcal{E}) \otimes \mathcal{F}) = 0, \text{ for } N \gg 0,\ i > \alpha(\mathcal{E})$$
$$H^i(X, \mathcal{E}^{\otimes N} \otimes \mathcal{F}) = 0, \text{ for } N \gg 0,\ i > \alpha_\otimes(\mathcal{E})$$

and

$$H^i(X, \mathcal{E}^{(p^N)} \otimes \mathcal{F}) = 0, \text{ for } N \gg 0,\ i > \phi(\mathcal{E})$$

for every locally free sheaf $\mathcal{F}$.

The next example will be taken up again in the last section. Suppose that $f : X \to X$ is an endomorphism of schemes, then we define the *f-amplitude* $\alpha_f(\mathcal{E}) = \alpha_P(\mathcal{E})$ with $P^n = (f^*)^n$. When $char k = p > 0$, and $F : X \to X$ is the absolute Frobenius, then α_F coincides with the Frobenius amplitude because $(F^*)^n\mathcal{E} = \mathcal{E}^{(p^n)}$.

We list a few basic properties of these invariants.

Lemma 2.1. $\alpha(\mathcal{E}) = \alpha(O_{\mathbb{P}(\mathcal{E})}(1)) = \alpha_\otimes(O_{\mathbb{P}(\mathcal{E})}(1))$.

Remark 2.2. *We are using the convention* $\mathbb{P}(\mathcal{E}) = \mathbf{Proj}(S^*(\mathcal{E}))$.

Proof. We have $\alpha(O_{\mathbb{P}(\mathcal{E})}(1)) = \alpha_\otimes(O_{\mathbb{P}(\mathcal{E})}(1))$ since the symmetric and tensor power powers of $O_{\mathbb{P}(E)}(1)$ coincide. The isomorphism

$$H^i(X, S^N(\mathcal{E}) \otimes \mathcal{F}) \cong H^i(\mathbb{P}(\mathcal{E}), O_{\mathbb{P}}(N) \otimes \pi^*\mathcal{F}) = H^i(\mathbb{P}(\mathcal{E}), O_{\mathbb{P}}(1)^{\otimes N} \otimes \pi^*\mathcal{F}) \tag{6}$$

where $\pi : \mathbb{P}(\mathcal{E}) \to X$ is the projection, implies $\alpha(\mathcal{E}) \leq \alpha(O_{\mathbb{P}(\mathcal{E})}(1))$. For the opposite inequality, it suffices to prove that $H^i(\mathbb{P}(E), O_{\mathbb{P}}(N) \otimes \mathcal{G}) = 0$ for all coherent $\mathcal{G}$ on $\mathbb{P}(E)$, $i > \alpha(\mathcal{E})$ and $N \gg 0$. We do this by descending i. Call a sheaf *special* if it isomorphic to a sheaf of the form $O_{\mathbb{P}}(j) \otimes \pi^*\mathcal{F}$. For special sheaves, the desired vanishing statement follows from (6). In general, observe that for any coherent sheaf $\mathcal{G}$ on $\mathbb{P}(E)$, we have a surjection $\pi^*\pi_*\mathcal{G}(M) \to \mathcal{G}(M)$ for $M \gg 0$. This is just a relative form of Serre's global generation theorem. Thus we have an exact sequence

$$0 \to \mathcal{G}' \to \mathcal{G}'' \to \mathcal{G} \to 0$$

where the middle sheaf is special. The vanishing of $H^i(O_{\mathbb{P}}(N) \otimes \mathcal{G})$ for $i > \alpha(\mathcal{E})$ now follows by descending induction. □

Lemma 2.3. *Fix an ample line bundle $O_X(1)$. Then the following are equivalent:*

(a) $\alpha_P(\mathcal{E}) \leq A$.
(b) *For any b there exists N_0 such that*

$$H^i(P^N(\mathcal{E})(b)) = 0$$

for all $i > A$, $N \geq N_0$.
(c) *For any coherent sheaf $\mathcal{F}$ there exists N_0 such that*

$$H^i(X, P^N(\mathcal{E}) \otimes \mathcal{F}) = 0$$

for all $i > A$, $N \geq N_0$.

Proof. To prove that (b) implies (c), we use the fact that any coherent sheaf $\mathcal{F}$ can be resolved as

$$0 \to \mathcal{F}_1 \to \bigoplus_i O_X(b_i) \to \mathcal{F} \to 0$$

Statement (c) can then be proven by descending induction on A. The other implications are automatic. □

Lemma 2.4. *If $\mathcal{E}$ is a vector bundle, then $\alpha(\mathcal{E}) = 0$ if and only if $\mathcal{E}$ is ample. If char $k = 0$, then $\alpha_\otimes(\mathcal{E}) = 0$ if and only if $\mathcal{E}$ is ample.*

Proof. This follows from [H, prop 3.3, section 5] and the previous lemma. □

Lemma 2.5. *If $\mathcal{E}$ is a vector bundle and n a positive integer, then*

$$\alpha_\otimes(\mathcal{E}^{\otimes n}) = \alpha_\otimes(\mathcal{E}).$$

Proof. Let $\mathcal{F}$ be a vector bundle. Then we can choose $N_0 > 0$ such that

$$H^i(X, \mathcal{E}^{\otimes nN+a} \otimes \mathcal{F}) = H^i(X, (\mathcal{E}^{\otimes n})^{\otimes N} \otimes (\mathcal{E}^{\otimes a} \otimes \mathcal{F})) = 0$$

for $i > \alpha_\otimes(\mathcal{E}^{\otimes n})$, $N > N_0$, $0 \leq a < n$. This shows that $\alpha_\otimes(\mathcal{E}^{\otimes n}) \geq \alpha_\otimes(\mathcal{E})$. For the opposite inequality, observe that

$$H^i((\mathcal{E}^{\otimes n})^{\otimes N} \otimes \mathcal{F}) = 0$$

for $i > \alpha(\mathcal{E})$ and $N \gg 0$. □

Corollary 2.6. *If $\mathcal{E}$ is a vector bundle and n a positive integer, then $\alpha(S^n(\mathcal{E})) \geq \alpha(\mathcal{E})$.*

Proof. This follows by applying the lemma to $O_{\mathbb{P}(\mathcal{E})}(1)$ and appealing to lemma 2.1. □

The following is a generalization of [A2, 2.4(4)]:

Lemma 2.7. *Let $f : X \to Y$ be a morphism of projective varieties. Suppose that $P^\bullet : Vect(X) \to Vect(X)$ and $\tilde{P}^\bullet : Vect(Y) \to Vect(Y)$ be a collection of operations which are compatible in the sense that $P^\bullet(f^*\mathcal{E}) = f^*\tilde{P}^\bullet(\mathcal{E})$. Then*

$$\alpha_P(f^*\mathcal{E}) \leq \alpha_{\tilde{P}}(\mathcal{E}) + d$$

where d is the maximum of the dimensions of the fibers.

Proof. Let $\mathcal{F}$ be a locally free O_X-module. We use the Leray spectral sequence

$$E_2^{ij} = H^i(Y, \tilde{P}^N(\mathcal{E}) \otimes R^j f_*\mathcal{F}) \Rightarrow H^{i+j}(X, P^N(f^*\mathcal{E}) \otimes \mathcal{F})$$

By assumption $E_2^{ij} = 0$ when $j > d$. Also $E_2^{ij} = 0$ for $i > \alpha_{\tilde{P}}(\mathcal{E})$, all j and $N \gg 0$, by lemma 2.3. Therefore

$$H^i(X, P^N(f^*\mathcal{E}) \otimes \mathcal{F}) = 0$$

for $i > \alpha_{\tilde{P}}(\mathcal{E}) + d$ and $N \gg 0$. □

Corollary 2.8. *If $X \to Y$ is a closed immersion, then $\alpha(\mathcal{E}|_X) \le \alpha(\mathcal{E})$.*

Corollary 2.9. *If $\mathcal{E} \to \mathcal{F}$ is a surjection of vector bundles, then $\alpha(\mathcal{F}) \le \alpha(\mathcal{E})$*

Proof. We have an inclusion $\mathbb{P}(\mathcal{F}) \subset \mathbb{P}(\mathcal{E})$ such that

$$O_{\mathbb{P}(\mathcal{E})}(1)|_{\mathbb{P}(\mathcal{F})} = O_{\mathbb{P}(\mathcal{F})}(1).$$

Now apply the previous corollary. □

There is a log version of amplitude, analogous to the notion of Frobenius amplitude $\phi(\mathcal{E}, D)$ of $\mathcal{E}$ relative to D given in [A2]. Let X be a smooth projective variety with a reduced normal crossing divisor D. Given a vector bundle $\mathcal{E}$ on X, define the *amplitude of $\mathcal{E}$ relative to D* by

$$\alpha(\mathcal{E}, D) = \min\{\alpha(S^n(\mathcal{E})(-D')) \mid n \in \mathbb{N}, 0 \le D' \le (n-1)D\}$$

It is perhaps more instructive to view this as the minimum of the amplitudes of the "vector bundles" $\mathcal{E}\langle -\Delta\rangle$ [L, 6.2A] as Δ ranges over strictly fractional effective $\mathbb{Q}$-divisors with support in D.

Lemma 2.10. *Let $\pi : \mathbb{P}(\mathcal{E}) \to X$ be the canonical projection, then $\alpha(\mathcal{E}, D) = \alpha(O_{\mathbb{P}(\mathcal{E})}(1), \pi^* D)$.*

Proof. This is immediate from the definition. □

Lemma 2.11. *Let $Y \to X$ be a morphism of projective varieties with Y smooth. Suppose that $D = \sum D_i$ is a divisor with normal crossings on Y, such that there exist $a_i \ge 0$ for which $L = O_Y(-\sum a_i D_i)$ is relatively ample. Then for any locally free sheaf $\mathcal{E}$ on X, $\alpha(f^*\mathcal{E}, D) \le \alpha(\mathcal{E})$.*

Proof. After replacing f by $\mathbb{P}(f^*\mathcal{E}) \to \mathbb{P}(\mathcal{E})$ and invoking the previous lemma, we can assume that $\mathcal{E}$ is a line bundle. Choose $n_0 > a_i$, and set $\mathcal{M} = f^*\mathcal{E}^{\otimes n_0} \otimes L$. Let $\mathcal{F}$ be a coherent sheaf on Y. Since L is relatively ample, the higher direct images of $\mathcal{F} \otimes L^{\otimes n}$ vanish for $n \gg 0$. The spectral sequence

$$H^a(\mathcal{E}^{\otimes n n_0} \otimes R^b f_*(\mathcal{F} \otimes L^{\otimes n})) \Rightarrow H^i(\mathcal{M}^{\otimes n} \otimes \mathcal{F})$$

yields the vanishing of the abutment for $i > \alpha(\mathcal{E})$ and $n \gg 0$. □

We define the generic amplitude of a vector bundle $\mathcal{E}$ on a projective variety Y by

$$\alpha_{gen}(\mathcal{E}) = \inf \alpha(r^*\mathcal{E}, Ex(r)),$$

where $r : Y' \to Y$ varies over birational maps from smooth varieties resolutions with normal crossing exceptional divisor $Ex(r)$. In contrast to $\alpha(\mathcal{E})$, this is a birational invariant.

Lemma 2.12. *Suppose that $\operatorname{char} k = 0$, and let $f : X \to Y$ be a morphism of projective varieties. Then*

$$\alpha_{gen}(f^*\mathcal{E}) \le \alpha(\mathcal{E}) + d$$

where d is the dimension of the generic fiber.

Proof. This can be deduced from lemma 2.11 in exactly the same way that corollary 2.8 was deduced from lemma 2.6 in [A2]. □

Recall that a vector bundle $\mathcal{E}$ is *d-ample* [So] if some $O_{\mathbb{P}(\mathcal{E})}(m)$ is base point free and dimensions of the fibers of the induced map

$$\phi_m : \mathbb{P}(\mathcal{E}) \to \mathbb{P}(H^0(O_{\mathbb{P}(\mathcal{E})}(m)))$$

are all less than or equal to d. An immediate consequence of the preceding lemma is:

Proposition 2.13 (Sommese)**.** *Suppose that some positive symmetric power $S^m(\mathcal{E})$ is globally generated. Then $\alpha(\mathcal{E}) \leq d$ if and only if $\mathcal{E}$ is d-ample.*

Proof. This is proven in [So, prop 1.7] under the blanket assumption that $k = \mathbb{C}$. For any field, when $\mathcal{E}$ is d-ample, the inequality $\alpha(\mathcal{E}) \leq d$ is a consequence of lemma 2.7 applied to $P = \mathbb{P}(\mathcal{E})$. Conversely, suppose that $S^m(\mathcal{E})$ and therefore that $O_{\mathbb{P}(\mathcal{E})}(m)$ is globally generated. Then for any coherent sheaf $\mathcal{F}$ supported on a closed fiber of ϕ_m,

$$H^i(P, \mathcal{F}) \cong H^i(P, \mathcal{F} \otimes O_{\mathbb{P}(\mathcal{E})}(mN)) = 0$$

for $i > \alpha(\mathcal{E})$ and $N \gg 0$. This implies that the dimension of the fiber, which coincides with its coherent cohomological dimension, is at most $\alpha(\mathcal{E})$.

□

Since the global generation assumption for $S^m(\mathcal{E})$ is built into the definition of d-ampleness, the above equivalence fails without it. We want to refine this to get estimates of amplitude under a weaker assumption. Consider the condition $G(\mathcal{E}, m, U)$ that for some m, the map $H^0(X, S^m(\mathcal{E})) \otimes O_U \to S^m(\mathcal{E})|_U$ is surjective for some open set $U \subseteq X$. Under this assumption, we get canonical morphisms

$$\phi_{m,U} : \mathbb{P}(\mathcal{E}|_U) \to \mathbb{P}(H^0(X, S^m(\mathcal{E})))$$

and

$$\psi_{m,U} : U \to Grass(H^0(S^m(\mathcal{E})), rank(S^m(\mathcal{E}))),$$

where $Grass(V, r)$ is the Grassmanian of r-dimensional quotients of V. These maps are related. If Q denotes the universal quotient bundle on the Grassmanian, $\psi_{m,U}$ induces a map $\mathbb{P}(\mathcal{E}|_U) \to \mathbb{P}(Q)$, and $\phi_{m,U}$ is composite of this and the projection $\mathbb{P}(Q) \to \mathbb{P}(H^0(X, S^m(\mathcal{E})))$.

Proposition 2.14. *Suppose that $\operatorname{char} k = 0$ and that $\mathcal{E}$ is a vector bundle on a smooth projective variety X such that $G(\mathcal{E}, m, U)$ holds.*

(1) *If the restriction of the universal bundle Q to $\overline{\psi_{m,U}(U)}$ is d-ample, then*
$$\alpha_{gen}(\mathcal{E}) \leq d + \dim X - \dim \psi_{m,U}(U)$$
(2) *If $\phi_{m,U}$ extends to a map $\overline{\phi}_{m,U} : Y \to \mathbb{P}(H^0(X, S^m(\mathcal{E})))$ of some nonsingular model of $\mathbb{P}(\mathcal{E})$ such that the dimensions of all the fibers of $\overline{\phi}_{m,U}$ are at most d, then*
$$\alpha(\mathcal{E}) \leq max(d, (rank(\mathcal{E}) - 1) + dim(X - U)).$$

We postpone the proof until after the next lemma.

Lemma 2.15. *Suppose $\operatorname{char} k = 0$. Let $\mathcal{L}$ be a line bundle over a projective variety X with at worst rational Gorenstein singularities. Suppose that $f : Y \to X$ is a desingularization with a globally generated sub-line bundle $\mathcal{M} \subset f^*\mathcal{L}$, and let $B \subset Y$ be the support of $\mathcal{L}/f_*(\mathcal{M})$. Then*

$$\alpha(\mathcal{L}) \leq max(\alpha(\mathcal{M}), \dim B)$$

Proof. Let $\mathcal{E}$ be a locally free sheaf on X. Set $\mathcal{E}' = \mathcal{E} \otimes \omega_X^{-1}$. By the Grauert-Riemenschneider vanishing theorem [GR]

$$\mathbb{R}f_*(\omega_Y \otimes \mathcal{M}^{\otimes n}) = f_*(\omega_Y \otimes \mathcal{M}^{\otimes n}) \tag{7}$$

This along with the projection formula yields a diagram with an exact rows

$$\begin{array}{ccccc}
H^i(\omega_Y \otimes \mathcal{M}^{\otimes n} \otimes f^*\mathcal{E}') & & & & \\
\| & & & & \\
H^i(\mathbb{R}f_*(\omega_Y \otimes \mathcal{M}^{\otimes n} \otimes f^*\mathcal{E}')) & & & & \\
\| & & & & \\
H^i(f_*(\omega_Y \otimes \mathcal{M}^{\otimes n}) \otimes \mathcal{E}') & \xrightarrow{r} & H^i(\omega_X \otimes \mathcal{L}^{\otimes n} \otimes \mathcal{E}') & \longrightarrow & H^i(Q \otimes \mathcal{E}')
\end{array}$$

where $Q = \omega_X \otimes \mathcal{L}^{\otimes n}/f_*(\omega_Y \otimes \mathcal{M}^{\otimes n})$. Since Q has support in B, r is surjective for $i > \dim B$, which implies

$$H^i(\mathcal{L}^{\otimes n} \otimes \mathcal{E}) = 0$$

for $n >> 0$ and $i > max(\alpha(\mathcal{M}), \dim B)$. □

Proof of proposition 2.14. The first inequality is a consequence of lemma 2.12 and proposition 2.13. The second follows by applying lemmas 2.1, 2.5, and 2.15 to $O_{\mathbb{P}(\mathcal{E})}(1)$. □

3. Subadditivity of amplitude

We continue the assumptions from the previous section.

Theorem 3.1. *If*

$$0 \to \mathcal{E}_1 \to \mathcal{E}_2 \to \mathcal{E}_3 \to 0$$

is an exact sequence of vector bundles, then $\alpha(\mathcal{E}_2) \leq \alpha(\mathcal{E}_1) + \alpha(\mathcal{E}_3)$.

Proof. Fix a sufficiently ample line bundle $O_X(1)$. Let $\mathcal{F}$ be a vector bundle. Choose $M \gg 0$ so that $\mathcal{F}(M)$ is regular, i.e.

$$\lambda(\mathcal{F}(M)) = 0. \tag{8}$$

From the definition of α and λ, observe that given a bundle $\mathcal{E}$ and an integer m, there exists an integer n_0 such that for $n \geq n_0$,

$$\lambda(S^n(\mathcal{E})(m)) \leq \alpha(\mathcal{E})$$

Therefore, we can find p_0 so that

$$\lambda(S^p(\mathcal{E}_1)(-M)) \leq \alpha(\mathcal{E}_1) \tag{9}$$

$$\lambda(S^p(\mathcal{E}_3)) \leq \alpha(\mathcal{E}_3) \tag{10}$$

for $p \geq p_0$. Choose $N \gg 0$ so that

$$\lambda(S^p(\mathcal{E}_i) \otimes \mathcal{F}(N)) = 0 \tag{11}$$

for $p < p_0$ and $i = 1, 3$. Finally choose $r_0 > p_0$ so that for $r \geq r_0$,

$$\lambda(S^{r-p}(\mathcal{E}_i)(-N)) \leq \alpha(\mathcal{E}_i) \tag{12}$$

when $p < p_0$ and $i = 1, 3$. We claim that for $r \geq r_0$

$$\mathcal{G} = S^p(\mathcal{E}_1) \otimes S^{r-p}(\mathcal{E}_3) \otimes \mathcal{F}$$

has level at most $\alpha(\mathcal{E}_1) + \alpha(\mathcal{E}_3)$. This follows by applying proposition 1.11 to a decomposition $\mathcal{G} = \mathcal{G}_1 \otimes \mathcal{G}_2 \otimes \ldots$, to obtain

$$\lambda(\mathcal{G}) \leq \sum \lambda(\mathcal{G}_i) \leq \alpha(\mathcal{E}_1) + \alpha(\mathcal{E}_3),$$

in the three cases:

I. If $p < p_0$, then take

$$\mathcal{G}_1 = S^{r-p}(\mathcal{E}_3)(-N),\ \mathcal{G}_2 = S^p(\mathcal{E}_1) \otimes \mathcal{F}(N)$$

Use (11) and (12).

II. If $r - p < p_0$ then take

$$\mathcal{G}_1 = S^p(\mathcal{E}_1)(-N),\ \mathcal{G}_2 = S^{r-p}(\mathcal{E}_3) \otimes \mathcal{F}(N)$$

Use (11) and (12).

III. If $p, r - p \geq p_0$, then take

$$\mathcal{G}_1 = S^p(\mathcal{E}_1)(-M),\ \mathcal{G}_2 = S^{r-p}(\mathcal{E}_3),\ \mathcal{G}_3 = \mathcal{F}(M)$$

Use (8), (9) and (10).

By [H2, III, exer. 5.16], there is a filtration $F^\bullet \subseteq S^r(\mathcal{E}_2)$ such that

$$Gr^p(S^r(\mathcal{E}_2)) \cong S^p(\mathcal{E}_1) \otimes S^{r-p}(\mathcal{E}_3)$$

Repeated application of lemma 1.1 and the above claim will imply that $S^r(\mathcal{E}_2) \otimes \mathcal{F}$ will have level at most $\alpha(\mathcal{E}_1) + \alpha(\mathcal{E}_3)$ for $r > r_0$. Lemma 1.7 will imply that $H^i(S^r(\mathcal{E}_2) \otimes \mathcal{F}) = 0$ for $i > \alpha(\mathcal{E}_1) + \alpha(\mathcal{E}_3)$ as required. $\square$

The following theorem is proved by the same method as above.

Theorem 3.2. *Given vector bundles $\mathcal{E}_1$ and $\mathcal{E}_3$, we have*

$$\alpha_\otimes(\mathcal{E}_1 \oplus \mathcal{E}_3) \leq \alpha_\otimes(\mathcal{E}_1) + \alpha_\otimes(\mathcal{E}_3)$$

$$\alpha_\otimes(\mathcal{E}_1 \otimes \mathcal{E}_3) \leq \alpha_\otimes(\mathcal{E}_3) + \alpha_\otimes(\mathcal{E}_3)$$

Proof. Set $\mathcal{E}_2 = \mathcal{E}_1 \oplus \mathcal{E}_3$. The proof that $\alpha_\otimes(\mathcal{E}_2) \leq \alpha_\otimes(\mathcal{E}_1) + \alpha_\otimes(\mathcal{E}_3)$ will follow the same outline as the proof of theorem 3.1. Let $\mathcal{F}$ be a vector bundle. Choose $M \gg 0$ so that $\mathcal{F}(M)$ is regular, i.e. $\lambda(\mathcal{F}(M)) = 0$. We can find p_0 so that

$$\lambda(\mathcal{E}_1^{\otimes p}(-M)) \leq \alpha_\otimes(\mathcal{E}_1)$$

$$\lambda(\mathcal{E}_3^{\otimes p}) \leq \alpha_\otimes(\mathcal{E}_3)$$

for $p \geq p_0$. Choose $N \gg 0$ so that the sheaves $\mathcal{E}_i^{\otimes p} \otimes \mathcal{F}(N)$ are regular for $p < p_0$ and $i = 1, 3$. Finally choose $r_0 > p_0$ so that for $r \geq r_0$, $\lambda(\mathcal{E}_i^{\otimes (r-p)}(-N)) \leq \alpha_\otimes(\mathcal{E}_i)$ for $p < p_0$. We can then argue that for $r \geq r_0$

$$\mathcal{G} = \mathcal{E}_1^{\otimes p} \otimes \mathcal{E}_3^{\otimes (r-p)} \otimes \mathcal{F}$$

has level at most $\alpha_\otimes(\mathcal{E}_1) + \alpha_\otimes(\mathcal{E}_3)$, by splitting it up into cases as above. Lemma 1.7 will then imply that for $i > \alpha_\otimes(\mathcal{E}_1) + \alpha_\otimes(\mathcal{E}_3)$, $H^i(\mathcal{G}) = 0$. Therefore $H^i(\mathcal{E}_2^{\otimes r} \otimes \mathcal{F}) = 0$, since $\mathcal{E}_2$ can be decomposed into a direct sum of sheaves of the form $\mathcal{E}_1^{\otimes p} \otimes \mathcal{E}_3^{\otimes r-p}$.

By the first inequality

$$H^i((\mathcal{E}_1 \otimes \mathcal{E}_3)^{\otimes N} \otimes \mathcal{F}) \subset H^i(\mathcal{E}_2^{\otimes 2N} \otimes \mathcal{F}) = 0$$

for $i > \alpha_\otimes(\mathcal{E}_1) + \alpha_\otimes(\mathcal{E}_3)$ and $N \gg 0$. This implies the second inequality $\alpha_\otimes(\mathcal{E}_1 \otimes \mathcal{E}_3) \leq \alpha_\otimes(\mathcal{E}_3) + \alpha_\otimes(\mathcal{E}_3)$. □

Proposition 3.3. *If char $k = 0$, then for any vector bundle of rank r, we have*

$$\alpha(\mathcal{E}) \leq \alpha_\otimes(\mathcal{E}) \leq \alpha(\mathcal{E}^{\oplus r!})$$

Equality holds if $\mathcal{E}$ is globally generated.

Proof. Since $S^n(\mathcal{E})$ is a direct summand of $\mathcal{E}^{\otimes n}$, the inequality $\alpha(\mathcal{E}) \leq \alpha_\otimes(\mathcal{E})$ holds. By standard representation theoretic machinery [FH], the tensor power $\mathcal{E}^{\otimes n}$ can be decomposed into a direct sum of Schur powers $\mathbb{S}^\lambda(\mathcal{E})$ for various partitions λ of n. Each $\mathbb{S}^\lambda(\mathcal{E})$ is a direct summand of some

$$S^{q_1}(\mathcal{E}) \otimes \dots S^{q_{r!}}(\mathcal{E}),$$

and therefore of

$$S^{q_1 + \dots q_{r!}}(\mathcal{E}^{\oplus r!}),$$

by [H, prop 5.1], with $\sum q_i = n$. Thus

$$H^i(S^{q_1}(\mathcal{E}) \otimes \dots S^{q_{r!}}(\mathcal{E}) \otimes \mathcal{F}) = 0$$

for $i > \alpha(\mathcal{E}^{\oplus r!})$ and $n \gg 0$. Therefore

$$H^i(\mathcal{E}^{\otimes n} \otimes \mathcal{F}) = 0$$

for $i > \alpha(\mathcal{E}^{\oplus r!})$ and $n \gg 0$, as required.

The last part follows from [So, cor 1.10] and proposition 2.13 □

Corollary 3.4. $\alpha_\otimes(\mathcal{E}) \leq r!\alpha(\mathcal{E})$.

4. q-convexity

In this section, we work over $\mathbb{C}$. By the GAGA theorem [S], over a compact base variety, we can replace algebraic vector bundles by the corresponding analytic bundles. We will denote a geometric vector bundle $\pi : E \to X$ with the standard font, and write $\mathcal{E}$ for its sheaf of holomorphic sections.

A complex manifold Z is called *strongly q-convex* (in the sense of Andreotti-Grauert) if there is a C^∞ exhaustion $\psi : E \to \mathbb{R}$ such that the Levi form, given locally by

$$L(\psi) = \left(\frac{\partial^2 \psi}{\partial z_i \partial \overline{z_j}} \right),$$

has at most $q-1$ nonpositive eigenvalues at all points $z \in Z$ outside a compact set. If $\mathcal{E}$ is a holomorphic vector bundle, we say that it is strongly q-convex if its total space E is. A useful criterion is the following:

Lemma 4.1. *Let $\mathcal{E}$ be a holomorphic vector bundle with a Hermitean metric over a compact complex manifold X. Let Θ be the curvature of the compatible connection viewed as a C^∞ section of $E^* \otimes \overline{E}^* \otimes T_X^* \otimes \overline{T_X}^*$. If $\Theta(\xi, \bar{\xi}, -, -)$ has fewer than q nonpositive eigenvalues for all $\xi \neq 0$ and all $x \in X$. Then $\mathcal{E}^*$ is strongly q-convex.*

Proof. The argument is indicated in [AG, p. 257]. Nevertheless, we outline it here since our notation is slightly different. Since the signs of the curvatures of $\mathcal{E}$ and $\mathcal{E}^*$ are opposite, it suffices to prove the dual statement that if $\Theta(\xi, \bar{\xi}, -, -)$ has fewer than q nonnegative eigenvalues then $\mathcal{E}$ is q-convex with respect to ψ, where $\psi : E \to \mathbb{R}$ is the square of the norm.

We can choose local coordinates z_i about x and a local frame e_α of $\mathcal{E}$ such that the metric is given by matrix $h_{\alpha\beta}(z_1,\ldots)$ satisfying

$$\frac{\partial h_{\alpha\beta}}{\partial z_i}|_x = \frac{\partial h_{\alpha\beta}}{\partial \bar{z}_i}|_x = 0.$$

In these coordinates, the curvature tensor at $(x,\xi)\in E$ is given by

$$\Theta_{\alpha\beta ij} = -\frac{\partial^2 h_{\alpha\beta}}{\partial z_i \partial \bar{z}_j}$$

[SS, p. 119], and

$$\psi = \sum h_{\alpha\beta} e_\alpha \bar{e}_\beta$$

The Levi form $L(\psi)$ at (x,ξ) is represented by the matrix

$$\begin{pmatrix} -\sum \Theta_{\alpha\beta ij}\xi_\alpha \bar{\xi}_\beta & 0 \\ 0 & h_{\alpha\beta} \end{pmatrix}$$

and the lemma follows. □

Theorem 4.2 (Grauert)**.** *If $\mathcal{E}^*$ is a strongly q-convex vector bundle over a smooth complex projective variety, then $\alpha(\mathcal{E}) \le q-1$.*

The result follows from [G, theorem 4.3], which was stated without proof, so we supply one here. First, we observe:

Proposition 4.3. *Suppose that $\pi : E \to X$ is a geometric holomorphic vector bundle over a compact complex manifold X, with associated locally free sheaf $\mathcal{E}$. Then for any line bundle $\mathcal{L}$ on X, there exists a filtration, indexed by $\mathbb{N}$, on $H^i(E,\pi^*\mathcal{L})$ such that*

$$Gr^\ell H^i(E,\pi^*\mathcal{L}) \cong H^i(X, S^\ell(\mathcal{E}^*)\otimes\mathcal{L})$$

Proof. The argument is a straightforward modification of the proof of [AG, prop 26]. Choose an open Leray covering $\{U_i\}$ of X which trivializes $\mathcal{E}$ and $\mathcal{L}$. Then a section σ of $\pi^*\mathcal{L}$ over $\pi^{-1}U_i$ can be written as an infinite series

$$\sum \xi_1^{n_1}\ldots\xi_r^{n_r}\otimes\lambda_{n_1\ldots n_r}$$

where $\lambda_{n_1\ldots n_r}$ is a section of $\mathcal{L}(U_i)$ and ξ_j are coordinates along the fibers of E. We filter this so that $\sigma\in F^\ell$ if $\lambda_{n_1\ldots n_r}=0$ for $\sum n_i<\ell$. We have a splitting

$$\pi_*\pi^*\mathcal{L}(U_i) = \pi^*\mathcal{L}(\pi^{-1}U_i) = P_\ell(\pi^{-1}U_i)\oplus F^\ell(\pi^{-1}U_i) \tag{13}$$

where $P_\ell(\pi^{-1}U_i)$ is the space of polynomials (i.e. finite sums) of degree at most ℓ. Therefore $F^\ell/F^{\ell+1}$ can be identified with the space of homogeneous polynomials of degree ℓ.

In more invariant language, $F^\bullet$ is the $\mathcal{I}$-adic filtration, where $\mathcal{I}$ is the ideal of the zero section of E. In particular, the filtration globalizes. We can give $H^i(\pi^*\mathcal{L})$ the induced filtration. Since the higher direct images $R^i\pi_*$, $i>0$, vanish for coherent sheaves, we can identify $H^i(\pi^*\mathcal{L})\cong H^i(\pi_*\pi^*\mathcal{L})$ with their filtrations. Then we have isomorphisms

$$Gr^\ell(\pi_*\pi^*\mathcal{L}) \cong \pi_* Gr^\ell(\pi^*\mathcal{L}) \cong \pi_*[(\mathcal{I}^\ell/\mathcal{I}^{\ell+1})\otimes\pi^*\mathcal{L}] \cong S^\ell(\mathcal{E}^*)\otimes\mathcal{L} \tag{14}$$

The local splittings (13) patch to give a global splitting of the image of the sequence

$$0\to\mathcal{I}^\ell\pi^*\mathcal{L}\to\pi^*\mathcal{L}\to\pi^*\mathcal{L}/\mathcal{I}^\ell\pi^*\mathcal{L}\to 0$$

under π_*. Therefore, we have a diagram with exact rows

$$\begin{tikzcd} & H^i(\mathcal{I}^\ell \pi^*\mathcal{L}) \arrow[r] \arrow[d, "\cong"] & H^i(\pi^*\mathcal{L}) \arrow[r] \arrow[d, "\cong"] & H^i(\pi^*\mathcal{L}/\mathcal{I}^\ell\pi^*\mathcal{L}) \arrow[d, "\cong"] & \\ 0 \arrow[r] & H^i(\pi_*\mathcal{I}^\ell \pi^*\mathcal{L}) \arrow[r] & H^i(\pi_*\pi^*\mathcal{L}) \arrow[r] & H^i(\pi_*\pi^*\mathcal{L}/\mathcal{I}^\ell\pi^*\mathcal{L}) \arrow[r] & 0 \end{tikzcd}$$

The proposition is a now consequence of (14). □

Proof of theorem. By the proposition,

$$\dim H^i(E, \pi^* O(j)) \geq \sum_{\ell=0}^{L} H^i(X, S^\ell(\mathcal{E}) \otimes O(j)), \quad \forall L$$

The left side is finite dimensional for $i \geq q$ and arbitrary j by [AG, theorem 14]. Therefore $H^i(S^\ell(\mathcal{E}) \otimes O(j)) = 0$ for large ℓ and $i \geq q$. This implies the theorem by lemma 2.3. □

5. Estimates on Frobenius amplitude

In positive characteristic, the Frobenius amplitude, or more briefly F-amplitude, of a bundle $\mathcal{E}$ on X was defined at the beginning of section 2. Over a field k of characteristic 0, the definition is via specialization; see [A2] for a detailed discussion. Given a pair $(X, \mathcal{E})$, a thickening of it is a flat family $(\tilde{X}, \tilde{\mathcal{E}})$ over the spectrum of a subring $A \subset k$ of finite type, such that the original pair is given by base change. Then the F-amplitude is defined so that $\phi(\mathcal{E}) \leq C$ if and only if there exists a thickening such that $\phi(\tilde{\mathcal{E}}|_{X_q}) \leq C$ for all closed points $q \in Spec\, A$. In the sequel, we will of often write $\mathcal{E}_q$ for $\tilde{\mathcal{E}}|_{X_q}$, and $p(q)$ for the characteristic of A/q.

We recall the notion of *F-semipositivity* of vector bundles from [A2, 3.4]. If $char\, k = p > 0$, $\mathcal{E}$ is F-semipositive if the regularities of $\mathcal{E}^{(p^n)}$ are bounded away from $+\infty$. If $char\, k = 0$, $\mathcal{E}$ is F-semipositive if there exists a thickening $(\tilde{X}, \tilde{\mathcal{E}})$ such that the restrictions $\tilde{\mathcal{E}}|_{X_q}$ are F-semipositive. The definition is a technical one. The key properties to keep in mind are that the notion is stable under pullbacks, and tensor product with a F-semipositive bundle does not increase ϕ [loc. cit.].

Moreover, a line bundle $\mathcal{L}$ is F-semipositive if and only if it is *arithmetically nef*, which means that either $char\, k > 0$ and $\mathcal{L}$ is nef in the usual sense, or there is a thickening $(\tilde{X}, \tilde{\mathcal{L}})$ such that all specializations $\mathcal{L}_q$ are nef [A2, 3.12], [K2]. The proof of equivalence is based on [Ke1, 1.5] which yields a stronger conclusion:

Lemma 5.1. *Suppose that $\mathcal{L}$ is an arithmetically nef line bundle on a projective variety X with ample line bundle $O(1)$. Then there exists a thickening $(\tilde{X}, \tilde{\mathcal{L}}, \tilde{O}(1))$ such that the regularities of $\{\mathcal{L}_q^m \mid q \in Spec\, A \text{ closed}, m \geq 0\}$ are bounded above.*

We will say that the locally free sheaf $\mathcal{E}$ is arithmetically nef if and only $O_{\mathbb{P}(E)}(1)$ is. By applying [L, 6.2.12, 6.2.16] to the fibers of a thickening, we obtain some useful criteria for this condition.

Proposition 5.2.

(1) *The class of arithmetically nef sheaves is stable under quotients, extensions and tensor products.*

(2) *If $f : Y \to X$ is a surjective map of projective varieties, then a locally free sheaf $\mathcal{E}$ on X is arithmetically nef if and only if $f^*\mathcal{E}$ is.*

(3) *A locally free sheaf $\mathcal{E}$ is arithmetically nef if and only if $S^n(\mathcal{E})$ is for some $n > 0$.*

Corollary 5.3. *$\mathcal{E}$ is arithmetically nef if some positive symmetric power is globally generated.*

Given a locally free sheaf $\mathcal{E}$ and a $\mathbb{Q}$-divisor D, it is convenient to say that a formal twist $\mathcal{E}\langle D\rangle$ [L, 6.2A] is arithmetically nef if $S^n(\mathcal{E})\otimes O(nD)$ is arithmetically for some $n > 0$ such that nD is integral. The choice of n is immaterial by above.

The key estimate on Frobenius amplitude is contained in the following theorem.

Theorem 5.4. *Let X be a smooth projective variety defined over a perfect field k of characteristic $p > 0$. Let $O_X(1)$ be a sufficiently ample line bundle, and let $\mathcal{E}, \mathcal{F}$ be locally free sheaves on X. If $\iota > \lambda(\mathcal{E}(-\dim X))$ and if $p^N \geq reg(\mathcal{F})$, then*

$$H^\iota(X, \mathcal{E}^{(p^N)} \otimes \mathcal{F}) = 0$$

Proof. Let $n = \dim X$, and fix $N > 0$ as above. Set $f' = F^N$ where $F : X \to X$ is the absolute Frobenius (which acts by identity on X as a set, and by pth powers on O_X). The endomorphism f' is not k-linear, but this can be rectified by changing the k-scheme structure. Let X' be the fibered product $X \times_{Speck} Spec\, k$ over the p^Nth power map of k. Then f' factors as

$$X \xrightarrow{f} X' \xrightarrow{g} X$$

where f is k-linear and $g : X' \to X$ is the natural map. The morphism f is the relative p^Nth Frobenius. Note that f is flat since X is smooth, and that g is an isomorphism of schemes because k is perfect. Therefore g^*, g_* induce isomorphisms on cohomology. Set $\mathcal{E}' = g^*\mathcal{E}$ and $O_{X'}(1) = g^*O_X(1)$, the latter is easily seen to be sufficiently ample.

Set

$$C^i = \begin{cases} \mathcal{R}_{-i} \boxtimes O_{X'}(i) & \text{if } 0 \leq i > -n-1 \\ ker[\mathcal{R}_n \boxtimes O_{X'}(-n) \to \mathcal{R}_{n-1} \boxtimes O_{X'}(-n+1)] & \text{if } i = -n-1 \end{cases}$$

where $\mathcal{R}_i$ is defined as in section 1. These fit into a resolution $C^\bullet$ of the structure sheaf of the diagonal Δ. In general, for locally free sheaves $\mathcal{E}_i$, $(\mathcal{E}_1 \boxtimes \mathcal{E}_2) \otimes C^\bullet$ is quasiisomorphic to $\delta_*(\mathcal{E}_1 \otimes \mathcal{E}_2)$, where $\delta : X' \to X' \times X'$ is the diagonal embedding. Consequently, $D^\bullet = (O_{X'}(-n) \boxtimes O_{X'}(n)) \otimes C^\bullet$ gives another resolution of the diagonal.

Let $\gamma : X \to X' \times X$ be the morphism for which $p_1 \circ \gamma = f$ and $p_2 \circ \gamma = id$, where p_i are the projections. Let Γ be transpose of the graph of f, i.e. the image of γ with its reduced structure. Then $(1 \times f)^{-1}\Delta = \Gamma$, and $(1 \times f)^*D^\bullet$ gives a resolution for O_Γ by flatness of $1 \times f$. Then we have a quasiisomorphism

$$(\mathcal{E}' \boxtimes \mathcal{F}) \otimes (1 \times f)^*D^\bullet \cong \gamma_*(f^*\mathcal{E}' \otimes \mathcal{F}) = \gamma_*(\mathcal{E}^{(p^N)} \otimes \mathcal{F}). \tag{15}$$

Thus, we can compute

$$H^i(X, \mathcal{E}^{(p^N)} \otimes \mathcal{F}) = H^i(X' \times X, \gamma_*(f^*\mathcal{E}' \otimes \mathcal{F})) \tag{16}$$

using this resolution.

For $a \geq -n$, we have

$$(\mathcal{E}' \boxtimes \mathcal{F}) \otimes (1 \times f)^*D^a \cong (\mathcal{E}' \otimes \mathcal{R}_{-a}(-n)) \boxtimes (\mathcal{F} \otimes O_X(p^N(n+a)))$$

Künneth's formula implies

$$H^{\iota-a}(\mathcal{E}'\otimes\mathcal{R}_{-a}(-n)\boxtimes\mathcal{F}(p^N(n+a))) = \bigoplus_{r+s=\iota-a} H^r(\mathcal{E}'\otimes\mathcal{R}_{-a}(-n))\otimes H^s(\mathcal{F}(p^N(n+a))) \tag{17}$$

We claim that the above cohomology groups vanish for $\iota > \lambda(\mathcal{E}(-n))$. We have $a \geq -n$. When $a = -n$, cohomology in degree $\iota - a > n$ is automatically zero. Therefore we can assume $0 \geq a > -n$. Now $r + s = \iota - a > \lambda(\mathcal{E}(-n))$ implies $r > \lambda(\mathcal{E}(-n))$ or $s > \lambda(\mathcal{E}(-n)) \geq 0$. In the first case, the right side of (17) vanishes by corollary 1.9. When $s > 0$, the second group on the right vanishes, since $p^N(a_n) \geq reg(\mathcal{F})$. Thus

$$H^{\iota-a}(\mathcal{E}' \otimes \mathcal{R}_{-a}(-n) \boxtimes \mathcal{F}(p^N(n + a))) = 0$$

as claimed.

Therefore

$$H^\iota(X, \mathcal{E}^{(p^N)} \otimes \mathcal{F}) = 0$$

for $\iota > \lambda(\mathcal{E}(-n))$ and $p^N \geq reg(\mathcal{F})$ by (16), the above claim, and the spectral sequence

$$E_1 = H^{\iota-a}((\mathcal{E}' \boxtimes \mathcal{F}) \otimes (1 \times f)^* D^a) \Rightarrow H^\iota(X' \times X, \gamma_*(f^*\mathcal{E}' \otimes \mathcal{F}))$$

□

Corollary 5.5. *Let X be a smooth projective variety defined over a perfect field k. Let $O_X(1)$ be sufficiently ample, and $\mathcal{E}$ a locally free sheaf on X. Then the Frobenius amplitude $\phi(\mathcal{E})$ satisfies*

$$\phi(\mathcal{E}) \leq \lambda(\mathcal{E}(-\dim X))$$

Proof. By specialization, we can assume that k is a perfect field of characteristic $p > 0$. The result is now an immediate consequence of the theorem. □

Even though the following corollary is special case of the next theorem, we give a short direct proof.

Corollary 5.6. *Let X be a smooth projective variety defined over a perfect field k, and let $\mathcal{L}$ be a line bundle on X. Then $\phi(\mathcal{L}) \leq \alpha(\mathcal{L})$ holds under one of the following assumptions*

(1) *$char\, k = p > 0$.*
(2) *$char\, k = 0$ and $\mathcal{L}$ is arithmetically nef.*

Proof. Let $O_X(1)$ be a sufficiently ample line bundle. For some $m_0 > 0$, we have

$$\lambda(\mathcal{L}^m(-\dim X)) \leq \alpha(\mathcal{L}^m) \leq \alpha(\mathcal{L}) \tag{18}$$

whenever $m > m_0$. The previous corollary yields

$$\phi(\mathcal{L}^m) \leq \lambda(\mathcal{L}^m(-\dim X))$$

and therefore

$$\phi(\mathcal{L}^m) \leq \alpha(\mathcal{L}).$$

When $char\, k = p > 0$, we can choose m to be a power p^N, then

$$\phi(\mathcal{L}^{p^N}) = \phi(\mathcal{L})$$

so we are done in this case.

Suppose $char\, k = 0$. Choose a thickening $(\tilde{X}, \ldots)$ of $(X, \ldots)$ over $Spec\, A$. By semicontinuity, there is an open set $U \subset Spec\, A$ such that (18) holds with $\mathcal{L}$ replaced by $\mathcal{L}_q$ on the left, for all closed points $q \in U$. It would follow that

$$\phi(\mathcal{L}_q^m) \leq \alpha(\mathcal{L})$$

Pick $q \in U$, and let $p = p(q)$. Choose $p^N \geq m$, then by [A2, 4.5] the F-semipositivity of $\mathcal{L}$ implies

$$\phi(\mathcal{L}_q) = \phi(\mathcal{L}_q^{p^N}) = \phi(\mathcal{L}_q^m \otimes \mathcal{L}_q^{p^N - m}) \leq \phi(\mathcal{L}_q^m)$$

and we are done in this case. □

We now come to our main result, which is a refinement of [A2, 6.1].

Theorem 5.7. *Let X be a smooth projective variety defined over a field k of characteristic 0. If $\mathcal{E}_1, \ldots \mathcal{E}_m$ are arithmetically nef vector bundles over X, then the Frobenius amplitude*

$$\phi(\mathcal{E}_1 \otimes \ldots \otimes \mathcal{E}_m) \leq \sum_i (rank(\mathcal{E}_i) + \alpha(\mathcal{E}_i)) - m$$

We set up the notation for the proof, which is modeled on the proof of [A2, 6.1]. Let $r_i = rank(\mathcal{E}_i)$. Choose a sufficiently ample line bundle $O_X(1)$, and a thickening $(\tilde{X}, \tilde{\mathcal{E}}_1, \ldots \tilde{\mathcal{E}}_m, \tilde{O}_X(1))$ of $(X, \mathcal{E}_1, \ldots \mathcal{E}_m, O_X(1))$ over $Spec\, A$. We can assume that the sheaves $\tilde{\mathcal{E}}_i$ are locally free. We have the following notation: $\mathbb{S}^\lambda$ is the Schur functor associated to a partition $\lambda = (\lambda_1, \lambda_2, \ldots)$, if $q \in Spec A$, then $\mathcal{E}_{h,q}$ is shorthand for $(\tilde{\mathcal{E}}_h)|_{X_q}$. Then [A2, theorem 6.2] gives a quasi-isomorphism between $\mathcal{E}_{h,q}^{(p(q))}$ and a Carter-Lusztig complex

(19)
$$CL(\mathcal{E}_{h,q})^\bullet := \mathbb{S}^{(p(q))}(\mathcal{E}_{h,q}) \to \mathbb{S}^{(p(q)-1,1)}(\mathcal{E}_{h,q}) \to \ldots \mathbb{S}^{(p(q)-r_h+1,1,\ldots 1)}(\mathcal{E}_{h,q}) \to 0$$

for any closed point $q \in Spec\, A$ with $p(q)$ large enough.

Lemma 5.8. *For any integer m, there exists N_0 and a nonempty open set $U \subseteq Spec\, A$ such that*

$$H^i(X_q, \mathbb{S}^{(N+1,1,\ldots 1)}(\mathcal{E}_{h,q}) \otimes \tilde{O}_{X_q}(m)) = 0$$

for all $i > \alpha(\mathcal{E}_h)$, $N \geq N_0$ and $q \in U$.

Proof. The argument is basically a modification of [loc. cit, 6.5]. To avoid excessive notation, we will suppress "q" and other decorations whenever it is clear from context what is meant. Let $\pi : Flag(\mathcal{E}_h) \to X$ be the flag bundle of $\mathcal{E}_h$. Set $\mathcal{M} = O_X(m)$ and $\mathcal{L} = O_{\mathbb{P}(\mathcal{E}_h)}(1)$. Let $\kappa = (1 \ldots 1)$ be a partition of length at most r_h. The map π factors through a natural map $\pi_1 : Flag(\mathcal{E}_h) \to \mathbb{P}(\mathcal{E}_h)$. There is a canonical ample line bundle L_κ on $Flag(\mathcal{E}_h)$ such that $\pi_* L_\kappa = \mathbb{S}^\kappa(\mathcal{E}_h)$, and more generally $\pi_*(\pi_1^* O_{\mathbb{P}(\mathcal{E}_h)}(N) \otimes L_\kappa) = \mathbb{S}^{(N+1,1,\ldots 1)}(\mathcal{E}_h)$ for $N \geq 0$. Kempf's vanishing theorem [J, II, 4.5], implies that the higher direct images of $\pi_1^* O_{\mathbb{P}(\mathcal{E}_h)}(N) \otimes L_\kappa$ under π and π_1 vanish. Therefore we have

$$\begin{aligned} H^i(X_q, \mathbb{S}^{(N+1,1,\ldots 1)}(\mathcal{E}_h) \otimes \mathcal{M}) &\cong H^i(Flag(\mathcal{E}_{h,q}), \pi_1^* \mathcal{L}^N \otimes L_\kappa \otimes \pi^* \mathcal{M}) \\ &\cong H^i(\mathbb{P}(\mathcal{E}_{h,q}), \mathcal{L}^N \otimes \mathcal{F}) \\ &\cong H^i(\mathbb{P}(\mathcal{E}_{h,q}), \mathcal{L}^{N_0} \otimes \mathcal{L}^{N-N_0} \otimes \mathcal{F}) \end{aligned}$$

where $\mathcal{F} = \pi_{1*}(L_k \otimes \mathcal{M})$ on $\mathbb{P}(\mathcal{E}_h)$. The regularities of $(\mathcal{L}^{N-N_0} \otimes \mathcal{F})|_{\tilde{X}_q}$ are bounded above for $N - N_0 \geq 0$ by lemma 5.1. Therefore, by lemma 1.13, there is N_0 such that the ith cohomology of $\mathcal{L}^{N_0} \otimes \mathcal{L}^{N-N_0} \otimes \mathcal{F}$ vanishes for $N \geq N_0$ and $i > \alpha(\mathcal{E}_h) = \alpha(\mathcal{L})$. □

Proof of theorem 5.7. By the last lemma, it follows that after shrinking $Spec\, A$, the levels

$$\lambda(\mathbb{S}^{(p(q)-j,1,\ldots 1)}(\mathcal{E}_{h,q})(-\dim X) \leq \alpha(\mathcal{E}_h)$$

for $j = 0, \ldots r_h - 1$. Therefore, by corollary 5.5, the F-amplitudes of the sheaves $\mathbb{S}^{(p(q)-j,1,\ldots 1)}(\mathcal{E}_{h,q})$ occurring in (19) are at most $\alpha(\mathcal{E}_h)$. Since F-amplitude is subadditive under tensor products [loc. cit, theorem 4.1], it follows that entries of

$$CL(\mathcal{E}_{1,q})^{\bullet} \otimes \ldots \otimes CL(\mathcal{E}_{m,q})^{\bullet} \tag{20}$$

have F-amplitude bounded by $\sum \alpha(\mathcal{E}_h)$. Moreover, the complex (20) is a length $\sum(rank(\mathcal{E}_h) - 1)$ resolution of

$$(\mathcal{E}_1 \otimes \ldots \otimes \mathcal{E}_m)_q^{(p(q))} = \mathcal{E}_{1,q}^{(p(q))} \otimes \ldots \otimes \mathcal{E}_{m,q}^{(p(q))}.$$

Therefore [loc. cit, theorem 2.5] implies the desired inequality

$$\phi((\mathcal{E}_1 \otimes \ldots \mathcal{E}_m)_q) = \phi((\mathcal{E}_1 \otimes \ldots \mathcal{E}_m)_q^{(p(q))}) \leq \sum \alpha(\mathcal{E}_h) + \sum(rank(\mathcal{E}_h) - 1).$$

□

The following is a generalization of [A2, theorem 6.7]. Let us say that a vector bundle $\mathcal{E}$ is arithmetically nef along a reduced divisor $D = \sum D_i$ if $\mathcal{E}\langle -\sum r_i D_i \rangle$ is arithmetically nef for some $0 \leq r_i < 1$.

Theorem 5.9. *Let X be a smooth projective variety defined over a field k of characteristic 0. Suppose that $\mathcal{E}$ is a locally free sheaf which is arithmetically nef along a reduced divisor D with normal crossings. Then*

$$\phi(\mathcal{E}, D) \leq \alpha(\mathcal{E}, D) + rank(\mathcal{E})$$

Proof. We show that for almost all q there exists a divisor $0 \leq G \leq (p(q) - 1)D$ such that

$$\lambda(\mathbb{S}^{(p(q)-j,1,\ldots 1)}(\mathcal{E}_q)(-\dim X)(-G)) \leq \alpha(\mathcal{E}, D)$$

This is will yield a bound on F-amplitudes of the entries of the Carter-Lusztig complex $CL(\mathcal{E}_q)^{\bullet}(-G)$ as above.

Let $P = \mathbb{P}(\mathcal{E})$, $\mathcal{L} = O_P(1)$ and fix a very ample line bundle H on P. By assumption, $\mathcal{L}^{\ell}(-D')$ are arithmetically nef for some $0 \leq D' \leq (\ell - 1)D$. We have

$$\alpha(\mathcal{E}, D) = \alpha(\mathcal{L}^M(-D''_h))$$

for some $M > 0$ and $0 \leq D'' \leq (M-1)D$. Fix a constant c that will be determined later. Then for $a \gg 0$,

$$H^i(P, \mathcal{L}^{aM+b}(-aD'') \otimes H^c) = 0$$

for all $i > \alpha(\mathcal{E}, D)$, $b \in \{0, \ldots \ell - 1\}$. The collection of exponents $N_b = aM + b$ is a complete set of representatives for $\mathbb{Z}/\ell\mathbb{Z}$. Set $F = aD''$. This satisfies $0 \leq F \leq (N_b - 1)D$. We can find a nonempty open set $U \subset Spec\, A$ such that

$$H^i(P_q, \mathcal{L}_q^{N_b}(-F) \otimes H^c) = 0$$

for all $i > \alpha(\mathcal{E}, D)$, b and closed $q \in U$.

Given $N \geq M = \max N_b$, choose b such that $N \equiv N_b \mod \ell$, and set

$$G(N) = F + \frac{N - N_b}{\ell} D'$$

Arguing as in lemma 5.8, we obtain that

$$H^i(X_q, \mathbb{S}^{(N+1,1,\ldots 1)}(\mathcal{E})(-G(N)) \otimes O(r)) = H^i(P_q, \mathcal{L}^{N_b}(-F) \otimes \mathcal{F}) \tag{21}$$

where

$$\mathcal{F} = \mathcal{F}(N, r) = (\mathcal{L}^\ell(-D'))^{(N-N_b)/\ell} \otimes \pi_{1*}(L_{(1,\ldots 1)} \otimes \pi^* O(r))$$

Since the regularities of the sheaves in $\{\mathcal{F}(N, r)_q \mid N \geq M, q \in U\}$ are bounded above by lemma 5.1, we can choose the initial constant $c \ll 0$ so that the cohomologies in (21) vanish for $i > \alpha(\mathcal{E}, D)$ and $r = -2\dim X, \ldots - \dim X$ by lemma 1.13. This will force the levels

$$\lambda(\mathbb{S}^{(p(q)-j,1,\ldots 1)}(\mathcal{E}_{h,q})(-G(p(q)))(-\dim X)) \leq \alpha(\mathcal{E}, D)$$

for $p(q) \geq max(M, rank(\mathcal{E}))$ as required. □

The following is a refinement of [A2, theorem 7.1]. The hypothesis about global generation of $S^{rN}(\mathcal{E}) \otimes \det(\mathcal{E})^{-N}$ below can be interpreted as a strong semistability condition, see [A1, p. 247].

Theorem 5.10. *Let $\mathcal{E}$ be a rank r vector bundle on a smooth projective variety X such that $S^{rN}(\mathcal{E}) \otimes \det(\mathcal{E})^{-N}$ and $\det(\mathcal{E})^N$ are globally generated for some $N > 0$ prime to char k. Then $\phi(\mathcal{E}) \leq \alpha(\det(\mathcal{E}))$.*

Proof. By specialization, it is enough to prove the theorem over a field with *char* $k = p > 0$. Since $\Sigma_N = S^{rN}(\mathcal{E}) \otimes \det(\mathcal{E})^{-N}$ is globally generated, $S^{rN}(\mathcal{E})$ is a quotient of $H^0(\Sigma_N) \otimes \det(\mathcal{E})^N$. Therefore

$$\alpha(S^{rN}(\mathcal{E})) \leq \alpha(H^0(\Sigma_N) \otimes \det(\mathcal{E})^N) \leq \alpha(\det(\mathcal{E})).$$

The first inequality follows from corollary 2.9 and the last from [So, cor 1.10] (the proof of which is characteristic free). This implies that $\alpha(\mathcal{E}) \leq \alpha(\det(\mathcal{E}))$ by corollary 2.6.

Fix a sufficiently ample line bundle $O_X(1)$, and choose $q = p^n \equiv 1\, (mod\, N)$. Then as in the proof of [A2, theorem 7.1], we can conclude that $\mathcal{E}^{(q)}$ is a direct summand of $S^q(\mathcal{E})$. Therefore

$$H^{i+j}(X, \mathcal{E}^{(q)} \otimes O_X(-\dim X - 1 - j)) = 0$$

for $i > \alpha(\det(\mathcal{E}))$ and $q \gg 0$ subject to the earlier constraint. In other words, we have shown

$$\lambda(\mathcal{E}^{(q)}(-\dim X)) \leq \alpha(\det \mathcal{E})$$

By corollary 5.5,

$$\phi(\mathcal{E}) = \phi(\mathcal{E}^{(q)}) \leq \alpha(\det(\mathcal{E}))$$

□

Corollary 5.11. *Suppose that $S^{rN}(\mathcal{E}) \otimes \det(\mathcal{E})^{-N}$ is globally generated for some $N > 0$ prime to char k, and that some power of $\det(\mathcal{E})$ is globally generated. Then $\mathcal{E}$ is F-semipositive.*

Proof. It is enough to check this in positive characteristic. Let $\mathcal{E}_n = \mathcal{E}^{(p^n)} \otimes O_X(1)$. Then $\mathcal{E}_n$ satisfies the hypothesis of the theorem with $\det(\mathcal{E}_n)$ ample for all n. Therefore $\mathcal{E}_n$ is F-ample (i. e. has F-amplitude 0). This implies that the regularities of $\mathcal{E}^{(p^n)}$ are bounded away from ∞. □

6. Vanishing theorems

In this section, we fix a smooth n dimensional projective variety X defined over a field k of characteristic 0. Applying [A2, cor 8.6] to theorems 5.7, 5.10, and 5.9 yields:

Theorem 6.1. *Suppose that*

(1) $\mathcal{E}_1, \ldots \mathcal{E}_m, \mathcal{F}$ *are locally free sheaves on* X *or ranks* $r_1, \ldots r_m, s$ *respectively,*
(2) $\mathcal{E}_1$ *is arithmetically nef along a reduced divisor with normal crossings* D*,*
(3) $S^{sN}(\mathcal{F}) \otimes \det(\mathcal{F})^{-N}$ *and* $\det(\mathcal{F})^N$ *are globally generated for some* $N > 0$.

Then

$$H^q(X, \Omega^p_X(\log D)(-D) \otimes \mathcal{E}_1 \otimes \ldots \otimes \mathcal{E}_m \otimes \mathcal{F}) = 0$$

for

$$p + q > n + \alpha(\mathcal{E}_1, D) + \alpha(\mathcal{E}_2) + \ldots \alpha(\mathcal{E}_m) + \sum r_i - m.$$

Corollary 6.2. *Suppose that* $D = D_1 + \ldots D_d$ *is a divisor with normal crossings and* $\mathcal{E}$ *a vector bundle such that* $\mathcal{E}$ *is arithmetically nef along* D*, and each* $\mathcal{E}|_{D_i}$ *is arithmetically nef along* $D'_i = D_{i+1} + \ldots D_d$*. Then*

$$H^q(X, \Omega^p_X \otimes \mathcal{E}) = 0$$

for

$$p + q \geq n + rank(\mathcal{E}) + max(\alpha(\mathcal{E}, D), \alpha(\mathcal{E}|_{D_1}, D'_1) - 1, \alpha(\mathcal{E}|_{D_2}, D'_2) - 1, \ldots);$$

in particular for

$$p + q \geq n + rank(\mathcal{E}) + \alpha(\mathcal{E})$$

Proof. For the last part, we note the inequality

$$max(\alpha(\mathcal{E}, D), \alpha(\mathcal{E}|_{D_1}, D'_1) - 1, \ldots) \leq \alpha(\mathcal{E}).$$

Tensoring $\mathcal{E}$ with the sequences

$$0 \to \Omega^p_X(\log D'_{i-1})(-D'_{i-1}) \to \Omega^p_X(\log D'_i)(-D'_i) \to \Omega^p_{D_i}(\log D'_i)(-D'_i) \to 0$$

and applying the theorem implies the vanishing statement. □

The special case of the above result, when $\mathcal{E}$ is d-ample (and $D = 0$), is due to Sommese and Shiffman [SS, cor. 5.20].

Corollary 6.3. *Suppose that* $\mathcal{E}$ *is a vector bundle such that its pullback under a birational map* $f : Y \to X$*, with* Y *smooth, is arithmetically nef along a normal crossing divisor. Then we have*

$$H^i(X, \omega_X \otimes \mathcal{E}) = 0$$

if $i \geq \alpha_{gen}(\mathcal{E}) + rank(\mathcal{E})$*, where* $\omega_X = \Omega^n_X$.

Using the method of Manivel, we can get a vanishing theorem for tensor powers. We start with a preliminary lemma.

Lemma 6.4. *Let $\mathcal{E}$ and $\mathcal{F}$ be a vector bundles on X with $\mathcal{F}$ F-semipositive. Let P be the m-fold fiber product $\mathbb{P}(\mathcal{E}) \times_X \ldots \times_X \mathbb{P}(\mathcal{E})$ with projections $p_i : P \to \mathbb{P}(\mathcal{E})$ and $\pi : P \to X$. Then for any sequence of positive integers $r, j_1, \ldots j_m$,*

$$H^q(P, \Omega_P^p \otimes p_1^* O(j_1) \otimes \ldots p_m^* O(j_m) \otimes \pi^* \det(\mathcal{E})^{\otimes r} \otimes \pi^* \mathcal{F}) = 0$$

for $p + q > \dim P + \alpha_\otimes(\mathcal{E}) = m(rank(\mathcal{E}) - 1) + \dim X + \alpha_\otimes(\mathcal{E})$.

Proof. Let

$$\mathcal{G} = S^{j_1}(\mathcal{E}) \otimes \ldots S^{j_m}(\mathcal{E}) \otimes (\det \mathcal{E})^{\otimes r}.$$

Then since $\mathcal{G}$ is a summand of a tensor power of $\mathcal{E}$, we have $\alpha_\otimes(\mathcal{G}) \leq \alpha_\otimes(\mathcal{E})$. Furthermore, $\alpha(O_{\mathbb{P}(\mathcal{G})}(1)) = \alpha(\mathcal{G}) \leq \alpha_\otimes(\mathcal{G})$ by proposition 3.3.

We have an embedding $P \to \mathbb{P}(\mathcal{G})$ such that $\alpha_{\mathbb{P}(\mathcal{G})}(1)$ restricts to $p_1^* O(j_1) \otimes \ldots p_m^* O(j_m) \otimes \pi^* \det(\mathcal{E})^{\otimes r}$. Therefore by corollary 2.8

$$\alpha(p_1^* O(j_1) \otimes \ldots \otimes \pi^* \det(\mathcal{E})^{\otimes r}) \leq \alpha_\otimes(\mathcal{E})$$

So by [A2, prop 3.10, theorem 4.5] and theorem 5.7

$$\phi(p_1^* O(j_1) \otimes \ldots \otimes \pi^* \det(\mathcal{E})^{\otimes r} \otimes \pi^* \mathcal{F}) \leq \alpha_\otimes(\mathcal{E})$$

Therefore [A2, cor 8.6] implies that

$$H^q(P, \Omega_P^p \otimes p_1^* O(j_1) \otimes \ldots p_m^* O(j_m) \otimes \pi^* \det(\mathcal{E})^{\otimes r} \otimes \pi^* \mathcal{F}) = 0$$

vanishes for

$$p + q > \dim P + \alpha_\otimes(\mathcal{E}) \geq \alpha(p_1^* O(j_1) \otimes \ldots p_m^* O(j_m) \otimes \pi^* \det(\mathcal{E})^{\otimes r})$$

□

Theorem 6.5. *Let $\mathcal{E}$ and $\mathcal{F}$ be a vector bundles on X. Suppose that $\mathcal{F}$ is F-semipositive. Then for any sequence of integers $k_1, \ldots k_\ell, j_1, \ldots j_m$*

$$H^q(X, \Omega_X^p \otimes S^{k_1}(\mathcal{E}) \otimes \ldots S^{k_\ell}(\mathcal{E}) \otimes \wedge^{j_1} \mathcal{E} \otimes \ldots \wedge^{j_m} \mathcal{E} \otimes (\det \mathcal{E})^{\ell+n-p} \otimes \mathcal{F}) = 0$$

whenever

$$p + q > n + \sum (rank(\mathcal{E}) - j_i) + \alpha_\otimes(\mathcal{E}).$$

Proof. The proof of theorem A given in [M, section 2.2] can be carried out almost word for word in the present context, with $\mathcal{L}$ replaced by $\mathcal{F}$ and the appeal to Kodaira-Akizuki-Nakano by the use of lemma 6.4.

□

Corollary 6.6. *If $\mathcal{E}$ is globally generated and $\mathcal{F}$ F-semipositive, then*

$$H^q(X, \Omega_X^p \otimes S^{k_1}(\mathcal{E}) \otimes \ldots S^{k_\ell}(\mathcal{E}) \otimes \wedge^{j_1} \mathcal{E} \otimes \ldots \wedge^{j_m} \mathcal{E} \otimes (\det \mathcal{E})^{\ell+n-p} \otimes \mathcal{F}) = 0$$

whenever

$$p + q > n + \sum (rank(\mathcal{E}) - j_i) + \alpha(\mathcal{E})$$

Proof. This follows from proposition 3.3. □

We can extend the above results to certain geometric variations of Hodge structure. Recall that a morphism $f : X \to Y$ of smooth projective varieties is called semistable if it is given local analytically (or étale locally) by

$$y_1 = x_1 x_2 \ldots x_{d_1},\ y_2 = x_{d_1+1} x_{d_1+2} \ldots x_{d_2},\ \ldots$$

It follows that the discriminant $D \subset Y$ and its preimage $E \subset X$ are reduced divisors with normal crossings. Let $\Omega^{\bullet}_{X/Y}(\log E/D)$ denote the relative logarithmic de Rham complex. Then

$$H = \mathbb{R}^h f_* \Omega^{\bullet}_{X/Y}(\log E/D)$$

forms part of a variation of Hodge structure. We have a Gauss-Manin connection

$$\nabla : H \to \Omega^1_Y(\log D) \otimes H,$$

and a Hodge filtration

$$F^a H = image[\mathbb{R}^h f_* \Omega^{\geq a}_{X/Y}(\log E/D) \to H],$$

satisfying Griffiths' transverality

$$\nabla(F^a H) \subset \Omega^1_Y(\log D) \otimes F^{a-1} H.$$

The connection ∇ is integrable, so it fits into a complex

$$\Omega^{\bullet}(H) = H \to \Omega^1_Y(\log D) \otimes H \to \Omega^2_Y(\log D) \otimes H \ldots$$

which, by Griffiths' transversality, is compatible with the filtration when suitably shifted. The associated graded complex $Gr^a(\Omega^{\bullet}(H))$ can be identified with the complex

$$\Omega^{\bullet}_Y(\log D) \otimes R^{h+\bullet} f_* \Omega^{a-\bullet}_{X/Y}(\log E/D),$$

where the differentials are given by cup product with the Kodaira-Spencer class of f. Using the same method as in the proof of [A2, cor. 8.6] but replacing [DI] by [I, theorem 4.7], we obtain:

Theorem 6.7. *With the above assumptions and notation, for any coherent sheaf $\mathcal{E}$ on Y,*

$$\mathbb{H}^{b+a}(Y, Gr^a(\Omega^{\bullet}(H)) \otimes \mathcal{E}) = 0$$

for $a + b \geq \dim Y + \phi(\mathcal{E})$.

Corollary 6.8. *If $\mathcal{E}$ is locally free, then*

$$\mathbb{H}^{b+a}(Y, Gr^a(\Omega^{\bullet}(H)) \otimes \mathcal{E}) = 0$$

for $a + b \geq \dim Y + rank(\mathcal{E}) + \alpha(\mathcal{E})$.

Taking a to be maximal yields the following Kollár type vanishing theorem:

Corollary 6.9. *If $\mathcal{E}$ is locally free, then*

$$H^i(Y, R^h f_* \omega_X \otimes \mathcal{E}) = 0$$

for $i \geq rank(\mathcal{E}) + \alpha(\mathcal{E})$.

We conjecture that the last corollary should hold without the semistability hypothesis on f. As an immediate consequence of the semistable reduction theorem [KKMS], we see that this is so when Y is a curve:

Corollary 6.10. *Corollary 6.9 holds for an arbitrary map when* $\dim Y = 1$.

We also conjecture that corollary 6.9 holds when $\alpha(\mathcal{E})$ is replaced by α_{gen}. The special case when $f = id$ is proved in corollary 6.3.

7. Varieties with endomorphisms

Fix an integer q. By a *q-endomorphism* of a smooth variety X, we mean a morphism $\phi : X \to X$ such that $\phi^* : NS(X) \otimes \mathbb{Q} \to NS(X) \otimes \mathbb{Q}$ acts by multiplication by q, where $NS(X)$ is the Neron-Severi group.

Example 7.1. *The morphism $\mathbb{P}^n \to \mathbb{P}^n$ given by*

$$[x_0, \dots x_n] \mapsto [x_0^q, \dots x_n^q]$$

is a q-endomorphism.

Example 7.2. *Let X be an Abelian variety, and let $\mu_{q,X} : X \to X$ be the multiplication by q. This is a q^2-endomorphism* [Mu2, p. 75].

Example 7.3. *The product of two varieties with q-endomorphisms has a q-endomorphism.*

Before giving the final example, we remind the reader that there is a recipe for building a toric variety from a fan Δ in a lattice N [D, F, O]. This is denoted by $T_N emb(\Delta)$ or $X(\Delta)$. This has an action of the torus $T_N = Hom(\check{N}, \mathbb{C}^*)$. A T_N-equivariant divisor D is determined by a piecewise linear real valued function on the support of Δ taking integer values on N [O, sect 2.1]. The collection of these forms an Abelian group $SF(\Delta)$. Moreover, when $X(\Delta)$ is complete, we have a surjective homomorphism $SF(\Delta) \to Pic(X(\Delta))$ [O, cor. 2.5].

Example 7.4. *Assume that $X = X(\Delta)$ is smooth and projective. For any q, multiplication by q on N induces a morphism of fans $\Delta \to \Delta$* [O, sect. 1.5]. *This induces an endomorphism of the associated toric variety $\mu_{q,X} : X \to X$. It is clear that $\mu_{q,X}$ acts by multiplication by q on $SF(\Delta)$, and this implies that $\mu_{q,X}$ is q-endomorphism. For the special case of $\mathbb{P}^n$ with its standard toric structure, $\mu_{q,X}$ coincides with the morphism f given in example 7.1.*

For line bundles theorem 5.7 extends to q-morphisms.

Lemma 7.5. *Suppose that $f : X \to X$ is a q-morphism of a projective variety with $q > 1$. For any line bundle $\mathcal{L}$ on X, $\alpha_f(\mathcal{L}) \le \alpha(\mathcal{L})$*

Proof. The set of numerically trivial line bundles $Pic^\tau(X)$ forms a bounded family. Therefore given a vector bundle $\mathcal{E}$, we can choose $N_0 > 0$ so that $H^i(X, \mathcal{L}^{\otimes q^N} \otimes \mathcal{E} \otimes \mathcal{M}) = 0$ for $N \ge N_0$, $i > \alpha(\mathcal{L})$ and $\mathcal{M} \in Pic^\tau(X)$. By assumption, $(f^N)^*\mathcal{L} = \mathcal{L}^{\otimes q^N} \otimes \mathcal{M}$ for some $\mathcal{M} \in Pic^\tau(X)$. Thus $H^i(X, (f^N)^*\mathcal{L} \otimes \mathcal{E}) = 0$ for $i > \alpha(\mathcal{L})$. □

Theorem 7.6. *Suppose that X is a smooth projective k-variety with a q-endomorphism $f : X \to X$, such that $q > 1$ is prime to the characteristic of k. Then for any vector bundle $\mathcal{E}$ on X,*

$$H^i(X, \Omega_X^j \otimes \mathcal{E}) = 0$$

whenever $i > \alpha_f(\mathcal{E})$.

Corollary 7.7. *Then for any line bundle*

$$H^i(X, \Omega_X^j \otimes \mathcal{L}) = 0$$

for $i > \alpha(\mathcal{L})$. In particular, this holds for $i > 0$ if $\mathcal{L}$ is ample.

For toric varieties, the last part of the corollary, where is $\mathcal{L}$ ample, is due to Danilov [D, 7.5.2] (and Bott for $\mathbb{P}^n$). For Abelian varieties, where Ω^1_X is trivial, the last part is also well known [Mu2, III-16].

Theorem 7.6 is an immediate consequence of the next two lemmas. The first lemma is just a formulation of a well known trick used in Frobenius splitting arguments [MR].

We call a coherent sheaf $\mathcal{F}$ *f-split*, if there exists a split injection $\mathcal{F} \to f_*\mathcal{F}$.

Lemma 7.8. *If $\mathcal{F}$ is f-split, then $H^i(X, \mathcal{F} \otimes \mathcal{E}) = 0$ for $i > \alpha_f(\mathcal{E})$.*

Proof. Using the projection formula, we have an injection

$$H^i(X, \mathcal{F} \otimes \mathcal{E}) \to H^i(X, (f^N)_*\mathcal{F} \otimes \mathcal{E}) = H^i(X, \mathcal{F} \otimes (f^N)^*\mathcal{E})$$

□

Lemma 7.9. *Suppose that X is a smooth projective k-variety with q-endomorphism $f : X \to X$, with $q > 1$ prime to the characteristic of k. Then Ω^j_X is f-split.*

Proof. A splitting of the natural map $\iota : \Omega^j_X \to f_*\Omega^j_X$ is given by the $(1/q^{\dim X})Tr$, where Tr is Grothendieck's trace [De, 4]. The identity $Tr \circ \iota = q^{\dim X}$ can checked on open set $U \subset X$ over which f is etale. □

This technique can be adapted to prove other results. As an example, we give a relative vanishing theorem.

Theorem 7.10. *Suppose that q is an integer relatively prime to the characteristic of k. Let*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\pi"] & X \arrow[d, "\pi"] \\ Y \arrow[r, "\phi"] & Y \end{tikzcd}$$

be a commutative diagram of projective varieties with X smooth and f and ϕ q-morphisms. Then for any vector bundle $\mathcal{E}$ on Y

$$H^i(Y, R^\ell\pi_*\Omega^j_X \otimes \mathcal{E}) = 0$$

whenever $i > \alpha_f(\mathcal{E})$. In particular, this holds for $i > 0$ if $\mathcal{E}$ is an ample line bundle.

Proof. The f-splitting of Ω^j_X induces a ϕ-splitting on $R^\ell\pi_*\Omega^j_X$. □

Corollary 7.11. *The conclusion of the theorem holds when $\pi : X \to Y$ is a toric morphism of toric varieties.*

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